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Uniform theory of inhomogeneous waveguide modes near cut-off

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Abstract. A uniform asymptotic method is used to calculate the modal eigenvalues for an inhomogeneous dielectric waveguide for modes near cut-off, thus extending previous work on uniform methods. The existence of a finite homogeneous cladding medium is properly accounted for, with arbitrary variations of core refractive index.

1. Introduction

In the preceding paper (Arnold 1980) a study was carried out of the construction of asymptotic approximations to the eigenvalue of the differential equation

$$d^2\phi/d\rho^2+(U^2-V^2f+\mu/\rho^2)\phi=0, \tag{1.1}$$

where

$$\mu=\tfrac{1}{4}-m^2, \qquad m\in\{0,1,2,\ldots\}, \tag{1.2}$$

f is an arbitrary analytic function of ρ^2 such that

$$f=0, \qquad \rho=0, \tag{1.3a}$$

$$f=1, \qquad \rho=1, \tag{1.3b}$$

V is a large parameter and U^2 is the eigenvalue which is required to be found. This differential equation arises in the theory of wave propagation on a dielectric cylinder having a radial variation of refractive index. Boundary conditions

$$\text{(i)} \qquad \phi\sim\rho^{m+1/2}, \qquad \rho\to 0, \tag{1.4a}$$

$$\text{(ii)} \qquad d\phi/d\rho=K\phi, \qquad \rho=1 \tag{1.4b}$$

have to be applied, where

$$K=\tfrac{1}{2}+W\mathrm{K}'_m(W)/\mathrm{K}_m(W), \tag{1.5}$$

$$W^2=V^2-U^2 \tag{1.6}$$

and $\mathrm{K}_m(W)$ and $\mathrm{K}'_m(W)$ are the modified Hankel function and its derivative respectively. By assuming that U^2 was not too close to V^2 and μ was small compared to V^2, it was possible to obtain an expression for the eigenvalue U^2 (as $V\to\infty$) by constructing an asymptotic approximation to ϕ which is valid on *all* $0\leq\rho\leq 1$. This is in contrast to the approach of Kurtz and Streifer (1969), who used matched piecewise-uniform asymptotic representations. A perturbation theory for quasi-quadratic f,

$$f=\rho^2(1+\epsilon g),$$

was outlined, where ϵ is a small parameter and g an arbitrary analytic function of ρ^2.

However, the case of near cut-off modes, when $U^2\sim V^2$, was not considered, and this case introduces complications in the determination of the eigenvalue which we wish to examine in detail here. In the JWKB analysis of wave propagation problems (Fröman and Fröman 1965) certain points, called turning points in quantum mechanics and caustics in geometrical optics, are known to play a crucial role; at such points JWKB theory fails to give a valid asymptotic representation. In this problem, the caustic is the zero of the leading order term in the coefficient of ϕ in equation (1.1) (as $V\to\infty$), that is to say a point ρ_1 at which

$$U^2=V^2f, \qquad \rho=\rho_1. \tag{1.7}$$

Since $f=1$ at $\rho=1$, it is apparent from (1.7) that, as $U^2\to V^2$, $\rho_1\to 1$. Thus there is a confluence of the caustic $\rho=\rho_1$ and the end-point $\rho=1$ at which the boundary condition (1.4b) has to be applied. This invalidates the procedure used previously (Arnold 1980), as it was there assumed that the caustic and the boundary point were sufficiently well separated to allow the use of the Liouville–Green approximation (Olver 1974) to approximate ϕ at the boundary, this approximation subsequently being obtained from the original uniform representation. This means that higher-order approximation functions (Airy functions) are required in order to approximate the eigenvalue, and we shall seek to obtain these also from a uniform representation.

To do this it is convenient to make some small changes to the original differential equation (1.1). For this purpose we use (1.6) to obtain

$$\mathrm{d}^2\phi/\mathrm{d}\rho^2+(V^2Q^2-W^2+\mu/\rho^2)\phi=0, \tag{1.8}$$

where

$$Q^2=1-f. \tag{1.9}$$

We now regard W^2 as an eigenvalue of (1.8) and assert that Q^2 satisfies the following conditions:

(i) $Q^2=1$ when $\rho=0$; (1.10a)

(ii) $Q^2=0$ when $\rho=1$; (1.10b)

(iii) Q^2 is an analytic function of ρ^2; (1.10c)

(iv) $V^2Q^2-W^2$ has one zero in $0\leq\rho\leq 1$; (1.10d)

(v) $V^2Q^2-W^2+\mu/\rho^2$ has two zeros in $0\leq\rho\leq 1$. (1.10e)

Furthermore, it is convenient to express the fact that U^2 is nearly equal to V^2 by allowing W^2 to be small;

$$\lambda=W^2/V^2\sim \mathrm{O}(V^{-\kappa}), \qquad \kappa>0 \tag{1.11}$$

is introduced as a *hypothesis*. This then ensures that the terms V^2Q^2 and W^2 have different asymptotic orders in V. The exponent κ in general depends on the boundary conditions, but in this case it transpires that $\kappa=\frac{2}{3}$ is correct; we shall simply assume that

$$\lambda\sim \mathrm{O}(V^{-2/3}).$$

2. Uniform asymptotic solution

In the preceding paper we showed how a Liouville transform

$$(z_1^2-z^2)(\mathrm{d}z/\mathrm{d}\rho)^2 = U^2/V^2-f \qquad (2.1a)$$

$$= Q^2-W^2/V^2 \qquad (2.1b)$$

could be constructed, such that

$$\int_0^{z_1} (z_1^2-z^2)^{1/2}\,\mathrm{d}z = \int_0^{\rho_1} \left(\frac{U^2}{V^2}-f\right)^{1/2} \mathrm{d}\rho \qquad (2.2a)$$

$$= \int_0^{\rho_1} \left(Q^2-\frac{W^2}{V^2}\right)^{1/2} \mathrm{d}\rho, \qquad (2.2b)$$

where ρ_1 is the zero of the integrand. The point z is given in terms of ρ by

$$\int_z^{z_1} (z_1^2-z'^2)^{1/2}\,\mathrm{d}z' = \int_\rho^{\rho_1} (Q^2-\lambda)^{1/2}\,\mathrm{d}\rho', \qquad (2.3)$$

where

$$\lambda = W^2/V^2. \qquad (2.4)$$

In this sequel we will construct this transform in an indirect way which has the advantage of giving an explicit expansion equivalent to (2.3) (which is implicit); evaluation of this expansion is quite simple when $\rho = 1$, which we need to apply the boundary conditions.

We begin by introducing the intermediate variable ζ:

$$(\zeta_0^2-\zeta^2)(\mathrm{d}\zeta/\mathrm{d}\rho)^2 = Q^2. \qquad (2.5)$$

In order that ζ and ρ be analytic functions of each other it is necessary to ensure that the zeros of both sides of equation (2.5) are consistent; therefore

$$\int_0^{\zeta_0} (\zeta_0^2-\zeta^2)^{1/2}\,\mathrm{d}\zeta = \int_0^1 Q\,\mathrm{d}\rho \qquad (2.6)$$

and

$$\frac{\pi}{4}\zeta_0^2 = \int_0^1 Q\,\mathrm{d}\rho. \qquad (2.7)$$

Next we suppose that

$$z = \zeta+\lambda\eta_1+\lambda^2\eta_2+O(\lambda^3). \qquad (2.8a)$$

This form for z suggests itself as an extension of the method used by Olver (1975) for the case $\mu = 0$.

If z is to be the Liouville transform (through ζ) of ρ, then η_1 and η_2 have to be chosen so that

$$(z_1^2-z^2)(\mathrm{d}z/\mathrm{d}\rho)^2 = Q^2-\lambda, \qquad (2.8b)$$

and therefore

$$(z_1^2-z^2)(\mathrm{d}z/\mathrm{d}\zeta)^2 = (Q^2-\lambda)(\mathrm{d}\rho/\mathrm{d}\zeta)^2 \qquad (2.9)$$

$$= \zeta_0^2-\zeta^2-\lambda(\mathrm{d}\rho/\mathrm{d}\zeta)^2 \qquad (2.10)$$

where (2.5) has been used. If we suppose further that

$$z_1^2=\alpha_0+\lambda\alpha_1+\lambda^2\alpha_2+O(\lambda^3), \tag{2.11}$$

then we may substitute (2.8*b*) and (2.11) in (2.10), and equate powers of λ to find

$$\alpha_0=\zeta_0^2=\frac{4}{\pi}\int_0^1 Q\,\mathrm{d}\rho, \tag{2.12a}$$

$$\zeta\eta_1-(\zeta_0^2-\zeta^2)\,\mathrm{d}\eta_1/\mathrm{d}\zeta=\tfrac{1}{2}[(\mathrm{d}\rho/\mathrm{d}\zeta)^2+\alpha_1], \tag{2.12b}$$

and so on for higher-order terms. Equation (2.12*b*) is a differential equation for η_1 which is easily solved:

$$\eta_1=\frac{1}{2(\zeta_0^2-\zeta^2)^{1/2}}\int_\zeta^{\zeta_0}\left[\left(\frac{\mathrm{d}\rho}{\mathrm{d}\zeta'}\right)^2+\alpha_1\right]\frac{\mathrm{d}\zeta'}{(\zeta_0^2-\zeta'^2)^{1/2}}. \tag{2.13}$$

The parameter α_1, as yet undetermined, may be chosen to make η_1 vanish at the origin. In fact, this ensures that η_1 has only odd powers of ζ in its expansion about $\zeta=0$, which is necessary in order that z and ζ be analytic functions of each other at $\zeta=0$ for any λ. Thus

$$\alpha_1\int_0^{\zeta_0}\frac{\mathrm{d}\zeta'}{(\zeta_0^2-\zeta'^2)^{1/2}}=-\int_0^{\zeta_0}\left(\frac{\mathrm{d}\rho}{\mathrm{d}\zeta'}\right)^2\frac{\mathrm{d}\zeta'}{(\zeta_0^2-\zeta'^2)^{1/2}}, \tag{2.14}$$

and using (2.5) on the right of (2.14) gives

$$\frac{\pi}{2}\alpha_1=-\int_0^1\frac{\mathrm{d}\rho}{Q}. \tag{2.15}$$

Similar considerations hold for α_2 and η_2, but, as we shall not use them, we need not dwell on them; the differential equation for η_2 is

$$\zeta\eta_2-(\zeta_0^2-\zeta^2)\frac{\mathrm{d}\eta_2}{\mathrm{d}\zeta}=(\zeta_0^2-\zeta^2)\left(\frac{\mathrm{d}\eta_1}{\mathrm{d}\zeta}\right)^2-\eta_1^2+\alpha_2+\frac{2\mathrm{d}\eta_1}{\mathrm{d}\zeta}(\alpha_1-2\zeta\eta_1), \tag{2.16}$$

and α_2 is given by

$$\frac{\pi}{2}\alpha_2=\rho_1'\alpha_1-\frac{1}{4}\int_0^1\left(1-\rho_1'\frac{\mathrm{d}Q^2}{\mathrm{d}\rho}\right)\frac{\mathrm{d}\rho}{Q^3}, \tag{2.17}$$

where

$$\rho_1'=\lim_{\lambda\to 0}\left(\frac{\mathrm{d}\rho_1}{\mathrm{d}\lambda}\right). \tag{2.18}$$

(Equation (2.17) is obtained by expanding (2.2*b*) in powers of λ. The integral exists because, as $\rho\to 1$, the term in brackets vanishes.) In principle, all higher-order terms subsumed under the $O(\lambda^3)$ terms could be obtained in this way. Instead, we make a small modification; we drop the term $O(\lambda^3)$ altogether from (2.8*b*), so that we have *exactly*

$$z=\zeta+\lambda\eta_1+\lambda^2\eta_2. \tag{2.19}$$

Then equation (2.10) will need to be modified to

$$(z_1^2-z^2)(\mathrm{d}z/\mathrm{d}\zeta)^2=\zeta_0^2-\zeta^2-\lambda(\mathrm{d}\rho/\mathrm{d}\zeta)^2+O(\lambda^3) \tag{2.20a}$$

and (2.8*b*) becomes

$$(z_1^2-z^2)(\mathrm{d}z/\mathrm{d}\rho)^2=Q^2-\lambda+\mathrm{O}(\lambda^3). \tag{2.20b}$$

Since ρ, ζ and z are all analytic functions of each other, the $\mathrm{O}(\lambda^3)$ terms are uniform on $0\leq\rho\leq 1$, and may be transferred to the left-hand side of (2.20*b*) to give

$$[z_1^2-z^2+\mathrm{O}(\lambda^3)](\mathrm{d}z/\mathrm{d}\rho)^2=Q^2-\lambda. \tag{2.21}$$

As previously, z_1 is given by (2.11). We may now absorb the $\mathrm{O}(\lambda^3)$ term in (2.11) into the corresponding term in (2.21), and define z_1 *exactly* by

$$z_1^2=\alpha_0+\lambda\alpha_1+\lambda^2\alpha_2. \tag{2.22}$$

This completes our analysis of the transformation of the independent variable.

Now we transform ϕ by

$$\phi=(\mathrm{d}z/\mathrm{d}\rho)^{-1/2}\Phi. \tag{2.23}$$

Then Φ satisfies

$$\mathrm{d}^2\Phi/\mathrm{d}z^2+\{V^2[z_1^2-z^2+\mathrm{O}(\lambda^3)]+(\mu/\rho^2)(\mathrm{d}\rho/\mathrm{d}z)^2+h\}\Phi=0, \tag{2.24}$$

where

$$h=-\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)^{-1/2}. \tag{2.25}$$

As in the preceding paper it can be shown that

$$\frac{1}{\rho^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)^2=\frac{1}{z^2}+\mathrm{O}(1) \tag{2.26}$$

and so, if $\mu\sim\mathrm{O}(1)$,

$$\frac{\mu}{\rho^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)^2=\frac{\mu}{z^2}+\mathrm{O}(1). \tag{2.27}$$

The symbol $\mathrm{O}(1)$ refers to a function of z which is analytic on $0\leq\rho\leq 1$ and is $\mathrm{O}(1)$ as $V\to\infty$.

Similarly, h is an $\mathrm{O}(1)$ analytic function, and $V^2\mathrm{O}(\lambda^3)\sim\mathrm{O}(1)$ if $\lambda\sim\mathrm{O}(V^{-2/3})$. All these terms may be collected together to give

$$\mathrm{d}^2\Phi/\mathrm{d}z^2+[V^2(z_1^2-z^2)+\mu/z^2+\delta]\Phi=0, \tag{2.28}$$

where

$$\delta\sim\mathrm{O}(1). \tag{2.29}$$

As before (Arnold 1980) we can show that, as $V\to\infty$,

$$\Phi\sim\Phi_0 \tag{2.30}$$

where

$$\mathrm{d}^2\Phi_0/\mathrm{d}z^2+[V^2(z_1^2-z^2)+\mu/z^2]\Phi_0=0, \tag{2.31}$$

and finally

$$\Phi\sim(\mathrm{d}z/\mathrm{d}\rho)^{-1/2}\Phi_0. \tag{2.32}$$

This approximation was justified by the methods of Lynn and Keller (1970) in the preceding paper.

The approximation (2.32) with Φ_0 given by (2.31) is identical to our previous one, but with the difference that we have a totally different representation for z as a function of ρ (equation (2.19) as opposed to equation (2.3)).

The solution of (2.31) is (Arnold 1980)

$$\Phi_0 = z^{m+1/2} \exp(-Vz^2/2) L_\nu^{(m)}(Vz^2), \tag{2.33}$$

and this solution satisfies the boundary condition (i) at the origin, as z and ρ are analytic functions of each other and $z \to 0$ as $\rho \to 0$. The parameter ν is defined by

$$2(2\nu + m + 1) = Vz_1^2. \tag{2.34}$$

A contour integral for $L_\nu^{(m)}(u)$ is (Arnold 1977†, 1980)

$$2^m \, \mathrm{e}^{-u/2} L_\nu^{(m)}(u) = \frac{1}{2\pi \mathrm{i}} (\mathrm{e}^{-\nu\pi\mathrm{i}} W_1 + \mathrm{e}^{\nu\pi\mathrm{i}} W_2) \tag{2.35}$$

with, for $j = 1, 2$,

$$W_j = \int_{C_j} \left(\frac{1+s}{1-s}\right)^{\nu+(m+1)/2} (1-s^2)^{(m-1)/2} \, \mathrm{e}^{-su/2} \, \mathrm{d}s. \tag{2.36}$$

The contour C_1 passes from $s = -1$ to $s = \infty \mathrm{e}^{\mathrm{i}\sigma} (\sigma > 0)$ and C_2 is the image of C_1 in the real axis, in the reverse direction.

Thus, we have established the same approximation as in the preceding paper, but with a different representation for z, facilitated by the assumed smallness of λ.

3. Non-uniform asymptotic solution

We have yet to determine the eigenvalue W^2, which is a parameter in ν through equations (2.11) and (2.34). This is determined through the boundary condition (1.4b) at $\rho = 1$, and to do this we seek a simpler approximation than (2.32); we approximate Φ_0 further in the vicinity of $\rho = 1$.

Let $z \to z_0$ as $\rho \to 1$. Then, from equation (2.21),

$$[z_0^2 - z_1^2 + \mathrm{O}(\lambda^3)](\mathrm{d}z/\mathrm{d}\rho)^2 = -\lambda \tag{3.1}$$

(since $Q^2 = 0$ at $\rho = 1$). This indicates that

$$z_0^2 - z_1^2 \sim \mathrm{O}(\lambda). \tag{3.2}$$

Thus, as $\lambda \to 0$ ($W^2 \to 0$), $z_0 \to z_1$. This is the confluence of boundary and caustic referred to in § 1, and it becomes clear that the point z_0 will lie in the vicinity of z_1 if λ is small enough. The Liouville–Green approximation to Φ_0 will fail in this case; in fact we might expect the correct approximation to be in terms of Airy functions. The Liouville transform of equation (2.31),

$$\Phi_0 = (\mathrm{d}\tau/\mathrm{d}z)^{-1/2} \Psi, \tag{3.3a}$$

$$-\tau(\mathrm{d}\tau/\mathrm{d}z)^2 = z_1^2 - z^2, \tag{3.3b}$$

† Several typographical errors occurred in this paper; see reference.

would lead us to expect an asymptotic approximation for Ψ (see Olver 1974),

$$\Psi \sim \Psi_0, \tag{3.4a}$$

where

$$\Psi_0 = A_1 \operatorname{Ai}(V^{2/3}\tau) + A_2 \operatorname{Bi}(V^{2/3}\tau) \tag{3.4b}$$

and Ai(X) and Bi(X) are the standard Airy functions (Abramowitz and Stegun 1965, Olver 1974). A_1 and A_2 are constants which we must find. To do this we note that (3.3) and (3.4) imply

$$\Phi_0 \sim (\mathrm{d}\tau/\mathrm{d}z)^{-1/2}(A_1 \operatorname{Ai}(V^{2/3}\tau) + A_2 \operatorname{Bi}(V^{2/3}\tau)), \tag{3.5}$$

and we try to extract such an approximation from the integral representation (2.36). As this is precisely the method used by Slater (1960) to approximate the Whitaker function, which is essentially the same as our function Φ_0, we need only state the results. By integrating (2.36) by the method of steepest descent (Chester *et al* 1957), expressing the result as a sum of Airy functions and comparing with (3.5), we obtain

$$A_1 = A_0 \cos(\nu\pi) \tag{3.6a}$$

and

$$A_2 = -A_0 \sin(\nu\pi). \tag{3.6b}$$

The constant A_0 may be set to unity because of the homogeneity of the differential equations.

By using (3.3a) and (3.4a) in the boundary condition (1.4b), we can obtain the eigenvalue equation

$$\mathrm{d}\Psi_0/\mathrm{d}\tau = K_0\Psi_0, \qquad \tau = \tau_0, \tag{3.7}$$

where

$$K_0 = \left(\frac{\mathrm{d}\tau}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\rho}\right)^{-1}\left[K + \frac{1}{2}\left(\frac{\mathrm{d}z}{\mathrm{d}\rho}\right)^{-1}\frac{\mathrm{d}^2 z}{\mathrm{d}\rho^2} + \frac{1}{2}\left(\frac{\mathrm{d}z}{\mathrm{d}\rho}\right)\left(\frac{\mathrm{d}\tau}{\mathrm{d}z}\right)^{-1}\frac{\mathrm{d}^2\tau}{\mathrm{d}z^2}\right] \tag{3.8}$$

and

$$\tau_0 = \lim_{\rho\to 1}(\tau). \tag{3.9}$$

Expressing Ψ_0 in terms of the Airy functions by equation (3.4b) and rearranging leads to

$$\tan(\nu\pi) = \frac{\operatorname{Ai}'(V^{2/3}\tau_0) - V^{-2/3}K_0 \operatorname{Ai}(V^{2/3}\tau_0)}{\operatorname{Bi}'(V^{2/3}\tau_0) - V^{-2/3}K_0 \operatorname{Bi}(V^{2/3}\tau_0)}, \tag{3.10}$$

where the prime implies differentiation with respect to the argument of the Airy functions and we have used (3.6) to evaluate the left-hand side. Equation (3.10) is to be solved in conjunction with equation (2.34) for ν (or for z_1, since equation (2.34) connects ν and z_1 directly).

In order to solve equation (3.10) asymptotically as $V \to \infty$, we need to estimate some of the parameters appearing in it. We already have equation (2.22) for z_1; in addition we require expressions for z_0, τ_0 and various derivatives of τ and z evaluated at $\rho = 1$. The principal method for finding these values is to allow $\rho \to 1$, $\zeta \to \zeta_0$ in (2.8), (2.12), (2.5) and (3.3*b*) or their differentiated forms. In this way we obtain the following expressions:

$$z_0^2=\alpha_0+\lambda(\alpha_1+\beta_1)+\mathrm{O}(\lambda^2), \tag{3.11}$$

$$z_0^2-z_1^2=\lambda\beta_1+\mathrm{O}(\lambda^2), \tag{3.12}$$

$$\tau_0=\lambda(2^{-2/3}\alpha_0^{-1/3}\beta_1)+\mathrm{O}(\lambda^2), \tag{3.13}$$

where α_0 and α_1 are given by (2.12*a*) and (2.15) respectively, and

$$\beta_1=2^{2/3}\alpha_0^{1/3}\lim_{\rho\to 1}\left[\left(-\frac{\mathrm{d}Q^2}{\mathrm{d}\rho}\right)^{-2/3}\right]. \tag{3.14}$$

The derivatives that we require are

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\rho}\right)^{-1}=\left(\frac{\mathrm{d}\tau}{\mathrm{d}\rho}\right)^{-1}=\left(\frac{\tau_0}{\lambda}+\mathrm{O}(\lambda^2)\right)^{1/2} \tag{3.15}$$

$$=(2^{-2/3}\alpha_0^{-1/3}\beta_1)^{1/2}+\mathrm{O}(\lambda). \tag{3.16}$$

Further expressions for the other derivatives in equation (3.8) can be obtained, but we shall show that they are in any case negligible to the order to which we are working.

We are now in a position to explain the purpose of the condition $\lambda\sim\mathrm{O}(V^{-2/3})$. When this holds, the argument of the Airy functions in equation (3.10) is

$$V^{2/3}\tau_0\sim\mathrm{O}(1), \tag{3.17}$$

by equation (3.13). If λ is allowed to be larger than $\mathrm{O}(V^{-2/3})$, the Airy functions may be replaced by their asymptotic expansions for large argument, in which case the right-hand side of (3.10) is exponentially small, and we are returned to the case we considered earlier (Arnold 1980). The above condition on λ is therefore a means of ensuring that the eigenvalue W^2 is small enough to justify separate treatment.

We now estimate the order of magnitude of the term K_0 in (3.10), given by equation (3.8). We have

$$K=\tfrac{1}{2}+W\mathrm{K}_m'(W)/\mathrm{K}_m(W),$$

and this form invites us to consider two possibilities:

(i) W is large, even though λ is small;

(ii) W is small.

Possibility (i) above can arise since

$$\lambda=W^2/V^2\sim\mathrm{O}(V^{-2/3}) \tag{3.18a}$$

and therefore

$$W\sim\mathrm{O}(V^{2/3}). \tag{3.18b}$$

Possibility (ii) may arise because it is obviously conceivable that, for some V, $W^2=0$. This case corresponds to cut-off of the propagating mode, and does not violate (3.18*a*).

If (i) above is true, then

$$K\sim-W+\tfrac{1}{2}+\mathrm{O}(W^{-1}) \tag{3.19}$$

$$\sim\mathrm{O}(V^{2/3}). \tag{3.20}$$

Therefore, as $V \to \infty$, the terms containing second derivatives in equation (3.8) are negligible, and we obtain

$$V^{-2/3}K_0 \sim O(1). \tag{3.21}$$

In fact, if we let

$$X = V^{2/3}\tau_0, \tag{3.22}$$

then equation (3.10) may be approximated by

$$\tan(\nu\pi) \sim \frac{\mathrm{Ai}'(X) + X^{1/2}\,\mathrm{Ai}(X)}{\mathrm{Bi}'(X) + X^{1/2}\,\mathrm{Bi}(X)} \tag{3.23}$$

when the precise form of K_0 is calculated. The value of λ satisfying the boundary condition at $\rho = 1$ is obtained by solving equations (3.23) and (2.34) for λ, by eliminating ν.

We postulate the existence of an expansion

$$\lambda = V^{-2/3}[\lambda_0 + \lambda_1 V^{-1/3} + O(V^{-2/3})]. \tag{3.24}$$

It turns out that λ_0 and λ_1 cannot be entirely independent of V, but are functions of the parameter

$$p = V^{-1/3}\Delta \tag{3.25}$$

where

$$\Delta = V - 2N/\alpha_0, \tag{3.26a}$$

$$N = 2q + m + 1, \qquad q \in \{0, 1, 2 \ldots\}. \tag{3.26b}$$

Nevertheless, since V and N can be regarded as independent of each other, we may also consider p and V to be mutually independent for the purpose of obtaining an asymptotic approximation for λ.

Since equation (3.23) is not changed if an integer q is subtracted from ν, we let

$$\nu = q + \theta, \qquad 0 \leq \theta < 1, \tag{3.27}$$

and, in the same spirit as equation (3.24), we suppose that

$$\theta = \theta_0 + \theta_1 V^{-1/3} + O(V^{-2/3}) \tag{3.28}$$

and

$$X = X_0 + X_1 V^{-1/3} + O(V^{-2/3}). \tag{3.29}$$

Then substitution of (3.27), (3.28), (3.29), (3.24) in (3.23) and (2.34), (using (2.22) to express z_1^2 as a series in powers of $V^{-1/3}$ like (3.24)) and elimination of ν by equating similar terms leads to

$$\lambda_0 = -\left(\frac{\alpha_0}{\alpha_1}\right)p = -\left(\frac{\alpha_0}{\alpha_1}\right)V^{-1/3}\Delta \tag{3.30}$$

and

$$\lambda_1 = \frac{1}{4\pi\alpha_1}\tan^{-1}\left(\frac{\mathrm{Ai}'(X_0) + X_0^{1/2}\,\mathrm{Ai}(X_0)}{\mathrm{Bi}'(X_0) + X_0^{1/2}\,\mathrm{Bi}(X_0)}\right) \tag{3.31}$$

where

$$X_0 = 2^{-2/3}\alpha_0^{-1/3}\beta_1\lambda_0. \tag{3.32}$$

Equations (3.30) and (3.31) are the final expressions for the coefficients in the expansion of λ, and, since $\lambda = W^2/V^2$, we may compute W^2 from these equations. The parameters α_0, α_1 and β_1 are given by (2.12*a*), (2.15) and (3.14) respectively. (It should be noted that, despite the minus sign in (3.30), λ_0 is positive because α_1 is negative.) The most useful expression we can derive is for W^2/V:

$$W^2/V = -\alpha_0\Delta/\alpha_1 + \lambda_1 + O(V^{-1/3}), \tag{3.33}$$

with λ_1 given by (3.31) and (3.32).

To obtain some insight into this expression, it is instructive to consider the case of exactly quadratic index variation,

$$f = \rho^2, \qquad Q^2 = 1 - \rho^2. \tag{3.34}$$

Then

$$\alpha_0 = -\alpha_1 = 1, \tag{3.35}$$

and we have

$$W^2/V = \Delta + \lambda_1. \tag{3.36}$$

The equivalent result neglecting the effect of the cladding medium, which we have shown elsewhere is accurate for modes not close to cut-off (Arnold 1980), would be

$$W^2/V = V - U^2/V = V - 2N \tag{3.37}$$

$$= \Delta. \tag{3.38}$$

This clearly shows that the term λ_1 in (3.33) is due to the presence of the finite cladding boundary. As Δ becomes sufficiently large, with V fixed, λ_0 and X_0 also have large values, and the term λ_1 becomes exponentially small through the behaviour of the Airy functions in equation (3.31). Since large Δ corresponds to large W^2, this is in exact agreement with our previous result (Arnold 1980), that for modes not close to cut-off the effect of the finite boundary is an exponentially small correction to the eigenvalue for an unbounded medium.

This completes our analysis of the case (i) above. We now repeat the analysis assuming (ii) above is true; W may not be large enough for (3.19) to hold.

In that case we find that

$$K \sim O(1) \tag{3.39}$$

and so

$$V^{-2/3}K_0 \sim O(V^{-2/3}). \tag{3.40}$$

Then it is clear that (3.10) may be approximated by

$$\tan(\nu\pi) \sim \frac{\mathrm{Ai}'(V^{2/3}\tau_0)}{\mathrm{Bi}'(V^{2/3}\tau_0)} + O(V^{-2/3}). \tag{3.41}$$

Since none of the other approximations made in case (i) are affected, we may carry the analysis through to obtain a modified expression for λ_1:

$$\lambda_1 = \frac{1}{4\pi\alpha_1}\tan^{-1}\left(\frac{\mathrm{Ai}'(X_0)}{\mathrm{Bi}'(X_0)}\right). \tag{3.42}$$

Equations (3.30) and (3.32) remain valid as written.

It is of some interest to calculate the value of V for which $W = 0$ exactly. In that case we find

$$K_0 \sim O(1) \tag{3.43}$$

and

$$\tau_0 = 0 \tag{3.44}$$

and so equation (3.10) becomes

$$\tan(\nu\pi) = \frac{\mathrm{Ai}'(0)}{\mathrm{Bi}'(0)} + O(V^{-2/3}) \tag{3.45}$$

$$= -3^{-1/2} + O(V^{-2/3}). \tag{3.46}$$

Hence

$$\nu = q - \tfrac{1}{6} + O(V^{-2/3}). \tag{3.47}$$

Also we have, since $\lambda = 0$,

$$z_1^2 = \alpha_0, \tag{3.48}$$

and so, equation (2.34) is equivalent to

$$2N - \tfrac{2}{3} + O(V^{-2/3}) = V\alpha_0, \tag{3.49}$$

and finally

$$V = (2N - \tfrac{2}{3})/\alpha_0 + O(V^{-2/3}) \tag{3.50}$$

is the final expression for the cut-off frequency, at which $W = 0$. When the index variation is exactly quadratic,

$$\alpha_0 = 1 \tag{3.51}$$

and (3.50) agrees with the correct result for this case (Arnold 1977). Since the error term in (3.50) is $O(V^{-2/3})$, this approximation is only valid for a mode whose cut-off frequency is high enough to ensure the smallness of this term.

4. Conclusions

It has here been shown how approximate expressions can be obtained for an eigenvalue problem arising in the theory of inhomogeneous dielectric waveguides; in particular, we have been concerned with the case of those modes which are close to cut-off, where the existence of finite boundaries has a significant effect on the asymptotic ($V \to \infty$) behaviour of the eigenvalues. The use of uniform asymptotic approximations to the solution has been demonstrated, and the further approximations necessary to solve the eigenvalue problem have been evaluated. It is hoped that these methods may eventually lead to a complete and systematic theory of propagation in waveguides having arbitrary variation of refractive index, under the assumption $V \to \infty$. Detailed calculations using these expressions are to be described elsewhere.

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References

Abramowitz M and Stegun I 1965 *Handbook of Mathematical Functions and Tables* (New York: Dover)
Arnold J M 1977 *IEE J. MOA* **1**, pp 203–8 (Errata 1978 *IEE J. MOA* **2** p 64)
—— 1980 *J. Phys. A: Math. Gen.* **13** 347–60
Chester C, Friedmann B and Ursell F 1957: *Proc. Camb. Phil. Soc.* **53** pp 599–611
Fröman N and Fröman P O 1965 *JWKB Approximation—Contributions to the Theory* (Amsterdam: North-Holland)
Kurtz C N and Streifer W 1969 *IEEE Trans. Micr. Theory and Tech.* MTT-17 pp 250–3
Lynn R Y S and Keller J B 1970 *Comm. Pure and App. Math.* **23** pp 379–408
Olver F W J 1974 *Asymptotics and Special Functions* (New York: Academic)
—— 1975 *Phil. Trans. R. Soc.* A **278** pp 137–74
Slater L J 1960 *Confluent Hypergeometric Functions* (Cambridge: CUP)